

Unified Lagrangian-Hamiltonian Formalism for Contact Systems

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We present a unified geometric framework for describing both the Lagrangian and Hamiltonian formalisms of contact autonomous mechanical systems, which is based on the approach of the pioneering work of R. Skinner and R. Rusk. This framework permits to skip the second order differential equation problem, which is obtained as a part of the constraint algorithm (for singular or regular Lagrangians), and is especially useful to describe singular Lagrangian systems. Some examples are also discussed to illustrate the method.

1. Introduction

In a seminal paper in 1983, R. Skinner and R. Rusk introduced a new framework for the dynamics of first-order autonomous mechanical systems which combined the Lagrangian and Hamiltonian formalisms^[47] into a single one. The aim of this formulation was to obtain a common framework for both regular and singular dynamics, describing simultaneously the Hamiltonian and the Lagrangian formulations of the dynamics. Over the years, Skinner–Rusk's framework was subsequently generalized in many directions. So, in [9] it was extended for explicit time-dependent systems using a jet bundle language, in [31] to other kinds of more general time-dependent singular differential equations, and in [3, 9] to first-order non-autonomous dynamical systems in general. In [17] the Skinner–Rusk formalism was used to study vakonomic mechanics and the comparison between the solutions of vakonomic and nonholonomic mechanics. The

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formalism was also extended to higherorder autonomous and non-autonomous mechanical systems,^[32,33,40,41] and it was also applied to control systems.^[2,16] Finally, in [8, 20, 22, 42, 45, 46, 48] the Skinner–Rusk model was developed for first and higher-order classical field theories and, in particular, it was used to describe different models of gravitational theories.^[11,12,24]

In recent years, there has been an increasing interest in the study of contact Hamiltonian and Lagrangian

systems.^[4,6,18,19,21,27,37] The essential tool is contact geometry,^[1,7,10,28] which has been used to describe dissipative systems^[13,23,38,44] and several other types of physical systems in thermodynamics, quantum mechanics, circuit theory, control theory, etc. (see for instance, [5, 30, 36, 43]). Recently, a generalization of contact geometry has been developed to describe field theories with dissipation.^[25,26]

In the contact setting the corresponding motion equations are obtained using the Herglotz principle instead of the Hamilton one.^[34,35] so that these dynamical systems do not enjoy conservative properties, but dissipative ones. The main difference between both variational principles is that, in the Herglotz variational principle, the action is defined by a non-autonomous ODE instead of an integral. Therefore, if we start with a Lagrangian function $L: TQ \times \mathbb{R} \to \mathbb{R}$ such that $L = L(q^i, v^i, z)$ using bundle coordinates, then the solutions to the dynamics obey the Herglotz equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z},$$

where $v^i = \dot{q}^i$, and they are sometimes called *generalized Euler-Lagrange equations*.

The contact Hamiltonian picture is obtained on the bundle $T^*Q \times \mathbb{R}$ just considering the canonical contact form $\eta = dz - \theta_o$, where $\theta_o = p_i dq^i$ (in bundle coordinates) is the canonical Liouville form on T^*Q . So, given a Hamiltonian function $H : T^*Q \times \mathbb{R} \to \mathbb{R}$, we can find a unique Hamiltonian vector field satisfying the equations

$$i(X_H)d\eta = dH - (\mathcal{R}(H))\eta, \quad i(X_H)\eta = -H,$$

where \mathcal{R} is the *Reeb vector field* characterized by the conditions

$$i(\mathcal{R})d\eta = 0, \quad i(\mathcal{R})\eta = 1.$$

The integral curves of $X_{\!H}$ satisfy the contact Hamiltonian equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\left(\frac{\partial H}{\partial q^i} + p_i\frac{\partial H}{\partial z}\right), \quad \frac{dz}{dt} = p_i\frac{\partial H}{\partial p_i} - H.$$

When the Lagrangian L is regular (in the usual sense) we can pass from the Lagrangian to the Hamiltonian picture by means of the corresponding Legendre transformation.

The aim of this paper is to extend the Skinner-Rusk formalism to contact dynamical systems (Section 3), now, carefully studying the dynamical equations of motion and the submanifold where they are consistent, and showing how the Lagrangian and Hamiltonian descriptions are recovered from this unified framework.

First, we define the *extended unified bundle* (also called the *extended Pontryagin bundle*) $W = TQ \times_Q T^*Q \times \mathbb{R}$. Then we consider a precontact form on W, which is just the pull-back of the canonical contact form on $T^*Q \times \mathbb{R}$. Finally, the Hamiltonian energy is constructed from a Lagrangian $L \in C^{\infty}(TQ \times \mathbb{R})$ by

$$\mathcal{H} = p_i v^i - L(q^j, v^j, z) \in C^{\infty}(\mathcal{W}).$$

The rest is just to apply a constraint algorithm to this precontact Hamiltonian system. One of the main interest in such formulation is that the SODE condition is obtained for free. If the Lagrangian is regular, we obtain the usual results when the dynamics are projected on the Lagrangian or the Hamiltonian side. In the singular case, the algorithm is properly connected (also by projection) with the corresponding Lagrangian and Hamiltonian constraint algorithms.

The paper is structured as follows. Section 2 is devoted to recall the main facts and results on contact Hamiltonian and Lagrangian dynamics. In Section 3 we develop the unified formalism and explain how the Lagrangian and Hamiltonian descriptions are recovered from it. Finally, in Section 4, we discuss several interesting examples of regular and singular systems.

All the manifolds are real, second countable and C^{∞} . The maps are assumed to be C^{∞} . Sum over repeated indices is understood.

2. Hamiltonian and Lagrangian Formalisms of Contact Systems

2.1. Contact Geometry and Contact Hamiltonian Systems

(See, for instance, [6, 27, 28, 37] for details).

Definition 1. Let M be a (2n + 1)-dimensional manifold. A contact form in M is a differential 1-form $\eta \in \Omega^1(M)$ such that $\eta \wedge (d\eta)^{\wedge n}$ is a volume form in M. Then (M, η) is said to be a contact manifold.

The fact that $\eta \wedge (d\eta)^{\wedge n}$ is a volume form induces a decomposition

 $TM = \ker d\eta \oplus \ker \eta \equiv \mathcal{D}^{\mathbb{R}} \oplus \mathcal{D}^{\mathbb{C}}.$

Proposition 1. If (M, η) is a contact manifold then there exists a unique vector field $\mathcal{R} \in \mathfrak{X}(M)$, which is called **Reeb vector field**, such that

 $i(\mathcal{R})d\eta = 0, \quad i(\mathcal{R})\eta = 1.$ (1)

This vector field generates the distribution \mathcal{D}^R , which is called the **Reeb** distribution.

In addition, for every point $p \in M$, there exist a chart $(U; q^i, p_i, z), 1 \le i \le n$, such that

$$\eta|_U = \mathrm{d}z - p_i \,\mathrm{d}q^i; \quad \mathcal{R}|_U = \frac{\partial}{\partial z}.$$

These are the *Darboux* or *canonical coordinates* of the contact manifold (M, η) .^[29]

The canonical model for contact manifolds is the manifold $T^*Q \times \mathbb{R}$. In fact, if z is the cartesian coordinate of \mathbb{R} , and $\theta_o \in \Omega^1(T^*Q)$ and $\omega_o = -d\theta_o \in \Omega^2(T^*Q)$ are the canonical forms in T^*Q , and $\pi_1 : T^*Q \times \mathbb{R} \to T^*Q$ is the canonical projection, then $\eta = dz - \pi_1^*\theta_o$ is a contact form in $T^*Q \times \mathbb{R}$, with $d\eta = \pi_1^*\omega_o$, and the Reeb vector field is $\mathcal{R} = \frac{\partial}{2}$.

the Reeb vector field is $\mathcal{R} = \frac{\partial}{\partial z}$. Given a contact manifold (M, η) , we have the $C^{\infty}(M)$ -module isomorphism

$$p: \quad \mathfrak{X}(M) \quad \to \quad \Omega^1(M) \\ X \quad \mapsto \quad i(X) \mathrm{d}\eta + (i(X)\eta)\eta$$

Theorem 1. If (M, η) is a contact manifold, for every $H \in C^{\infty}(M)$, there exists a unique vector field $X_H \in \mathfrak{X}(M)$ such that

$$i(X_H)d\eta = dH - (\mathcal{R}(H))\eta, \quad i(X_H)\eta = -H.$$
(2)

Then, the integral curves $\mathbf{c} : I \subset \mathbb{R} \to M$ of X_H are the solutions to the equations

$$i(\mathbf{c}')d\eta = (dH - (\mathcal{R}(H))\eta)\circ\mathbf{c}, \quad i(\mathbf{c}')\eta = -H\circ\mathbf{c},$$
 (3)

where $c':I\subset \mathbb{R} \to TM$ is the canonical lift of the curve c to the tangent bundle TM.

Definition 2. The vector field X_H is the contact Hamiltonian vector field associated to H and the Equations (2) and (3) are the contact Hamiltonian equations for this vector field and its integral curves, respectively. The triple (M, η, H) is said to be a contact Hamiltonian system.

Taking Darboux coordinates (q^i, p_i, z) , the contact Hamiltonian vector field is

$$X_{H} = \frac{\partial H}{\partial p_{i}} \frac{\partial}{\partial q^{i}} - \left(\frac{\partial H}{\partial q^{i}} + p_{i} \frac{\partial H}{\partial z}\right) \frac{\partial}{\partial p_{i}} + \left(p_{i} \frac{\partial H}{\partial p_{i}} - H\right) \frac{\partial}{\partial z};$$

and its integral curves $\mathbf{c}(t) = (q^i(t), p_i(t), z(t))$ are solutions to the dissipative Hamilton Equations (3) which are

$$\dot{q}^{i} = \frac{\partial H}{\partial p_{i}}, \quad \dot{p}_{i} = -\left(\frac{\partial H}{\partial q^{i}} + p_{i}\frac{\partial H}{\partial z}\right), \quad \dot{z} = p_{i}\frac{\partial H}{\partial p_{i}} - H.$$
 (4)

Remark 1. The contact Hamiltonian Equations (2) are equivalent to

$$\mathcal{L}(X_H)\eta = -(\mathcal{R}(H))\eta, \quad i(X_H)\eta = -H,$$

and also to

$$\flat(X_H) = \mathrm{d}H - (\mathcal{R}(H) + H)\eta.$$

Furthermore, Equations (2) can be written without making use of the Reeb vector field \mathcal{R} , as follows: consider the open set $U = \{p \in M; H(p) \neq 0\}$ and the 2-form $\Omega = -H d\eta + dH \wedge \eta$ on U. A vector field $X_H \in \mathfrak{X}(U)$ is the contact Hamiltonian vector field if, and only if,

$$i(X_H)\Omega = 0, \quad i(X_H)\eta = -H$$

On the open set *U*, a path $\mathbf{c} : I \subset \mathbb{R} \to M$ is an integral curve of the contact Hamiltonian vector field X_H if, and only if, it is a solution to

 $i(\mathbf{c}')\Omega = 0, \quad i(\mathbf{c}')\eta = -H \circ \mathbf{c}.$

Remark 2. When some of the conditions stated in Definition 1 do not hold, η is said to be a *precontact structure* and (M, η) is a *precontact manifold* (then the map \flat is not an isomorphism) and (M, η, H) is called a *precontact Hamiltonian system*. Then, the Hamiltonian equations are not necessarily compatible everywhere on M and a suitable *constraint algorithm* must be implemented in order to find a *final constraint submanifold* $P_f \hookrightarrow M$ (if it exists) where there are Hamiltonian vector fields $X_H \in \mathfrak{X}(M)$, tangent to P_f (which are not necessarily unique) solutions to the Hamiltonian equations on P_f . Furthermore, for precontact manifolds, Reeb vector fields are not uniquely determined but, if (M, η, H) is a precontact Hamiltonian system, the constraint algorithm and the final dynamics are independent on the Reeb chosen. (See [18] for a deeper analysis on all these topics).

2.2. Contact Lagrangian Systems

(See [13, 15, 18, 27] for details).

Let *Q* be an *n*-dimensional manifold and the bundle $TQ \times \mathbb{R}$ with canonical projections

$$z: \mathrm{T}Q \times \mathbb{R} \to \mathbb{R}, \quad \tau_1: \mathrm{T}Q \times \mathbb{R} \to \mathrm{T}Q, \quad \tau_0: \mathrm{T}Q \times \mathbb{R} \to Q \times \mathbb{R}.$$

Natural coordinates in $TQ \times \mathbb{R}$ are denoted (q^i, v^i, z) .

As a product manifold, we can write $T(TQ \times \mathbb{R}) = (T(TQ) \times \mathbb{R}) \bigoplus_{TQ \times \mathbb{R}} (TQ \times T\mathbb{R})$, so any operation that can act on tangent vectors to TQ can act on tangent vectors to $TQ \times \mathbb{R}$. In particular, the vertical endomorphism of T(TQ) and the Liouville vector field on TQ yield a *vertical endomorphism* $\mathcal{J} : T(TQ \times \mathbb{R}) \to T(TQ \times \mathbb{R})$ and a *Liouville vector field* $\Delta \in \mathfrak{X}(TQ \times \mathbb{R})$ (this is the Liouville vector field of the vector bundle structure defined by τ_0). In natural coordinates, their local expressions are

$$\mathcal{J} = \frac{\partial}{\partial v^i} \otimes \mathrm{d} q^i$$
, $\Delta = v^i \frac{\partial}{\partial v^i}$.

Let $\mathbf{c} : \mathbb{R} \to Q \times \mathbb{R}$ be a path, with $\mathbf{c} = (\mathbf{c}_1, \mathbf{c}_0)$. The *prolongation* of \mathbf{c} to $TQ \times \mathbb{R}$ is the path

$$\tilde{\mathbf{c}} = (\mathbf{c}_1', \mathbf{c}_0) : \mathbb{R} \to \mathrm{T}Q \times \mathbb{R}$$
,

where \mathbf{c}'_1 is the velocity of \mathbf{c}_1 . The path $\tilde{\mathbf{c}}$ is said to be *holonomic*. A vector field $\Gamma \in \mathfrak{X}(\mathrm{T}Q \times \mathbb{R})$ is said to satisfy the *second-order condition* (for short: it is a soDE) when all of its integral curves are holonomic. In coordinates, if $\mathbf{c}(t) = (c^i(t), z(t))$, then

$$\tilde{\mathbf{c}}(t) = \left(c^{i}(t), \frac{dc^{i}}{dt}(t), z(t)\right).$$

and the local expression of a SODE is

$$\Gamma = \nu^i \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial \nu^i} + g \frac{\partial}{\partial z}.$$

So, in coordinates a sob ${\tt sob} {\tt f}$ defines a system of differential equations of the form

$$\frac{\mathrm{d}^2 q^i}{\mathrm{d} t^2} = f^i(q,\dot{q},z), \quad \frac{\mathrm{d} z}{\mathrm{d} t} = \mathrm{g}(q,\dot{q},z).$$

A vector field $\Gamma \in \mathfrak{X}(TQ \times \mathbb{R})$ is a sobe if, and only if, $\mathcal{J}(\Gamma) = \Delta$.

Definition 3. A Lagrangian function is a function $L : TQ \times \mathbb{R} \to \mathbb{R}$. The Lagrangian energy associated with L is the function $E_L :=$

 $\Delta(L) - L \in \mathcal{C}^{\infty}(\mathcal{T}Q \times \mathbb{R}).$

The Cartan forms associated with L are defined as

$$\theta_L = {}^t \mathcal{J} \circ dL \in \Omega^1(\mathbb{T}Q \times \mathbb{R}), \quad \omega_L = -d\theta_L \in \Omega^2(\mathbb{T}Q \times \mathbb{R}).$$

The contact Lagrangian form is

$$\eta_L = \mathrm{d}z - \theta_L \in \Omega^1(\mathrm{T}Q \times \mathbb{R});$$

it satisfies that $d\eta_L = \omega_L$.

The couple (TQ \times \mathbb{R} , L) is a contact Lagrangian system.

In natural coordinates in $TQ \times \mathbb{R}$ we have

$$\begin{split} \eta_{L} &= \mathrm{d}z - \frac{\partial L}{\partial v^{i}} \,\mathrm{d}q^{i} \;, \\ \mathrm{d}\eta_{\mathcal{L}} &= -\frac{\partial^{2} L}{\partial z \partial v^{i}} \mathrm{d}z \wedge \mathrm{d}q^{i} - \frac{\partial^{2} L}{\partial q^{i} \partial v^{i}} \mathrm{d}q^{i} \wedge \mathrm{d}q^{i} - \frac{\partial^{2} L}{\partial v^{j} \partial v^{i}} \mathrm{d}v^{j} \end{split}$$

Now, we define the *Legendre map* associated with a Lagrangian *L* as the fiber derivative of *L*, considered as a function on the vector bundle $\tau_0 : TQ \times \mathbb{R} \to Q \times \mathbb{R}$; that is, the map $\mathcal{F}L : TQ \times \mathbb{R} \to T^*Q \times \mathbb{R}$ given by

$$\mathcal{F}L(\mathbf{v}_q, z) = \big(\mathcal{F}L(\cdot, z)(\mathbf{v}_q), z\big),$$

where $L(\cdot, z)$ is the Lagrangian with z fixed. Its local expression in natural coordinates is

$$\mathcal{F}L^*z = z, \quad \mathcal{F}L^*q^i = q^i, \quad \mathcal{F}L^*p_i = \frac{\partial \mathcal{L}}{\partial v^i}.$$

Remark 3. The Cartan forms can also be defined as $\theta_L = \mathcal{F}L^*(\pi_1^*\theta_o)$ and $\omega_L = \mathcal{F}L^*(\pi_1^*\omega_o)$.

Proposition 2. Given a Lagrangian L, then the Legendre map $\mathcal{F}L$ is a local diffeomorphism if, and only if, $(TQ \times \mathbb{R}, \eta_L)$ is a contact manifold.

The conditions in the proposition mean that the Hessian matrix $(W_{ij}) = (\frac{\partial^2 L}{\partial v^i \partial v^j})$ is everywhere nonsingular.

 $\wedge dq^i$,

Definition 4. A Lagrangian function L is said to be **regular** if the equivalent conditions in Proposition 2 hold. Otherwise, L is called a **singular** Lagrangian. In particular, L is said to be **hyperregular** if $\mathcal{F}L$ is a global diffeomorphism.

A singular Lagrangian is **almost-regular** if: (i) $P_1 = \mathcal{F}L(TQ \times \mathbb{R})$ is a closed submanifold of $T^*Q \times \mathbb{R}$, (ii) $\mathcal{F}L$ is a submersion onto its image, (iii) for every $v_q \in TQ \times \mathbb{R}$, the fibres $\mathcal{F}L^{-1}(\mathcal{F}L(v_q))$ are connected submanifolds of $TQ \times \mathbb{R}$.

Remark 4. As a result of the preceding definitions and results, every regular contact Lagrangian system has associated the contact Hamiltonian system ($TQ \times \mathbb{R}, \eta_L, E_L$).

Given a regular contact Lagrangian system (T $Q \times \mathbb{R}$, *L*), from (1) we have that the *Reeb vector field* $\mathcal{R}_L \in \mathfrak{X}(TQ \times \mathbb{R})$ for this system is uniquely determined by the relations

 $i(\mathcal{R}_L) \mathrm{d}\eta_L = 0, \quad i(\mathcal{R}_L)\eta_L = 1,$

and its local expression is

$$\mathcal{R}_{L} = \frac{\partial}{\partial z} - W^{ji} \frac{\partial^{2} L}{\partial z \partial v^{j}} \frac{\partial}{\partial v^{i}},$$

where (W^{ji}) is the inverse of the Hessian matrix, namely $W^{ji}W_{ik} = \delta_{i}^{j}$.

Definition 5. Let $(TQ \times \mathbb{R}, L)$ be a contact Lagrangian system.

The contact Euler–Lagrange equations for a holonomic curve \tilde{c} : $I \subset \mathbb{R} \to TQ \times \mathbb{R}$ are

$$i(\tilde{\mathbf{c}}')\mathrm{d}\eta_L = \left(\mathrm{d}E_L - (\mathcal{R}_L(E_L))\eta_L\right)\circ\tilde{\mathbf{c}}, \quad i(\tilde{\mathbf{c}}')\eta_L = -E_L\circ\tilde{\mathbf{c}} , \qquad (5)$$

where $\tilde{c}' : I \subset \mathbb{R} \to T(TQ \times \mathbb{R})$ denotes the canonical lifting of \tilde{c} to $T(TQ \times \mathbb{R})$.

The contact Lagrangian equations for a vector field $X_L \in \mathfrak{X}(\mathrm{T}Q \times \mathbb{R})$ are

$$i(X_L)\mathrm{d}\eta_L = \mathrm{d}E_L - (\mathcal{R}_L(E_L))\eta_L, \quad i(X_L)\eta_L = -E_L. \tag{6}$$

A vector field which is a solution to these equations is called a **contact Lagrangian vector field** (it is a contact Hamiltonian vector field for the function E_T).

Remark 5. In the open set $U = \{p \in M; \mathcal{H}(p) \neq 0\}$, the above equations can be stated equivalently as

$$i(\tilde{\mathbf{c}}')\Omega_L = 0, \quad i(\tilde{\mathbf{c}}')\eta_L = -E_L \circ \tilde{\mathbf{c}},$$

and

$$i(X_L)\Omega_L = 0, \quad i(X_L)\eta_L = -E_L$$

where $\Omega_L = -E_L d\eta_L + dE_L \wedge \eta_L$.

In natural coordinates, for a holonomic curve $\tilde{c}(t) = (q^i(t), \dot{q}^i(t), z(t))$, the contact Euler-Lagrange Equations (5) are

$$\dot{z} = L$$
, (7)

$$\frac{\partial^2 L}{\partial v^j \partial v^i} \ddot{q}^j + \frac{\partial^2 L}{\partial q^j \partial v^i} \dot{q}^j + \frac{\partial^2 L}{\partial z \partial v^i} \dot{z} - \frac{\partial L}{\partial q^i} = \frac{d}{dt} \left(\frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial z} \frac{\partial L}{\partial v^i};$$
(8)

meanwhile, for a vector field $X_L \in \mathfrak{X}(TQ \times \mathbb{R})$, if *L* is a regular Lagrangian, then X_L is a sode which is called the *Euler–Lagrange vector field* associated with *L* and whose integral curves are the Euler–Lagrange Equations (7) and (8). The local expression of this Euler–Lagrange vector field is

$$\begin{split} X_L &= L \frac{\partial}{\partial z} + \nu^i \frac{\partial}{\partial q^i} \\ &+ W^{ik} \bigg(\frac{\partial L}{\partial q^k} - \frac{\partial^2 L}{\partial q^j \partial \nu^k} \nu^j - L \frac{\partial^2 L}{\partial z \partial \nu^k} + \frac{\partial L}{\partial z} \frac{\partial L}{\partial \nu^k} \bigg) \frac{\partial}{\partial \nu^i}. \end{split}$$

Remark 6. If *L* is singular, although $(TQ \times \mathbb{R}, \eta_L)$ is not strictly a contact manifold, but a precontact one, and hence the Reeb vector field is not uniquely defined, it can be proved that the Lagrangian Equations (6) are independent on the Reeb vector field used (see [18]). Then, solutions to the Lagrangian equations are not necessarily sope and, in order to obtain the Euler–Lagrange Equations (8), the condition $\mathcal{J}(X_L) = \Delta$ must be added to the above Lagrangian equations. Furthermore, these equations are not necessarily compatible everywhere on $TQ \times \mathbb{R}$ and a suitable *constraint algorithm* must be implemented in order to find a *final constraint submanifold* $S_f \hookrightarrow TQ \times \mathbb{R}$ (if it exists) where there are sope vector fields $X_L \in \mathfrak{X}(TQ \times \mathbb{R})$, tangent to S_f , which are (not necessarily unique) solutions to the above equations on S_f . All these problems have been studied in detail in [18].

Remark 7. In the (hyper)regular case we have that $\mathcal{F}L$ is a diffeomorphism between $(TQ \times \mathbb{R})$ and $(T^*Q \times \mathbb{R})$, and $\mathcal{F}L^*\eta = \eta_L$. Furthermore, there exists (maybe locally) a function $H \in C^{\infty}(T^*Q \times \mathbb{R})$ such that $\mathcal{F}L^*H = E_L$; then we have the contact Hamiltonian system $(T^*Q \times \mathbb{R}, \eta, H)$, for which $\mathcal{F}L_*\mathcal{R}_{\mathcal{L}} = \mathcal{R}$. Then, if $X_H \in \mathfrak{X}(T^*Q \times \mathbb{R})$ is the contact Hamiltonian vector field associated with H, we have that $\mathcal{F}L_*X_L = X_H$.

In the almost-regular case we have the submanifold $j_1 : P_1 = \mathcal{F}L(\mathrm{T}Q \times \mathbb{R}) \hookrightarrow \mathrm{T}^*Q \times \mathbb{R}$, and $\mathcal{F}L^*\eta = \eta_L$. Then there exists a function $H_1 \in \mathbb{C}^{\infty}(P_1)$ such that $\mathcal{F}L^*H_1 = E_L$, and we have the precontact Hamiltonian system (P_1, η_1, H_1) , where $\eta_1 = j_1^*\eta$. The corresponding (precontact) Hamilton equations are not necessarily compatible everywhere on P_1 and a *constraint algorithm* must be implemented in order to find a *final constraint submanifold* $P_f \hookrightarrow P_1$ (if it exists) where there are vector fields $X_{H_1} \in \mathfrak{X}(P_1)$, tangent to P_f (which are not necessarily unique) solutions to the above equations on P_f . This algorithm and the equivalence between the Lagrangian and the Hamiltonian description of these precontact systems are also studied in [18].

3. Unified Formalism

3.1. Unified Bundle: Precontact Canonical Structure

For a contact dynamical system the *configuration space* is $Q \times \mathbb{R}$, where Q is an *n*-dimensional manifold, with coordinates (q^i, z) .

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Then, consider the bundles T*Q* × \mathbb{R} and T* *Q* × \mathbb{R} with canonical projections

$$\begin{aligned} \tau_1 &: \mathrm{T}Q \times \mathbb{R} \to \mathrm{T}Q, \quad \tau_0 &: \mathrm{T}Q \times \mathbb{R} \to Q \times \mathbb{R} \\ \pi_1 &: \mathrm{T}^*O \times \mathbb{R} \to \mathrm{T}^*O, \quad \pi_0 &: \mathrm{T}^*O \times \mathbb{R} \to O \times \mathbb{R} . \end{aligned}$$

with natural coordinates (q^i, v^i, z) and (q^i, p_i, z) adapted to the bundle structures. We denote by dz the volume form in \mathbb{R} , and its pull-backs to all the manifolds. Let $\theta_o \in \Omega^1(\mathbb{T}^*Q)$ and $\omega_o = -d\theta_o \in \Omega^2(\mathbb{T}^*Q)$ be the canonical forms of \mathbb{T}^*Q whose local expressions are $\theta_o = p_i dq^i$ and $\omega_o = dq^i \wedge dp_i$; and denote $\theta := \pi_1^* \theta_o$ and $\omega := \pi_1^* \omega_o$.

Definition 6. We define the **extended unified bundle** (also called the **extended Pontryagin bundle**)

$$\mathcal{W} = \mathrm{T}Q \times_{O} \mathrm{T}^{*}Q \times \mathbb{R} ,$$

which is endowed with the natural submersions

$$\begin{split} \rho_1 &: \mathcal{W} \to \mathrm{T} Q \times \mathbb{R} \text{ , } \rho_2 : \mathcal{W} \to \mathrm{T}^* Q \times \mathbb{R} \text{, } \rho_0 : \mathcal{W} \to Q \times \mathbb{R} \text{,} \\ z &: \mathcal{W} \to \mathbb{R} \text{.} \end{split}$$

The natural coordinates in \mathcal{W} are (q^i, v^i, p_i, z) .

Definition 7. We say that a path $\sigma : \mathbb{R} \to W$ is holonomic in W if the path $\rho_1 \circ \sigma : \mathbb{R} \to TQ \times \mathbb{R}$ is holonomic.

A vector field $X \in \mathfrak{X}(W)$ is said to satisfy the second-order condition in W (for short: it is a SODE in W) when all of its integral curves are holonomic in W.

In coordinates, a holonomic path in \mathcal{W} is expressed as

$$\boldsymbol{\sigma} = \left(\sigma_1^i(t), \frac{d\sigma_1^i}{dt}(t), \sigma_{2i}(t), \sigma_0(t)\right),$$

and a sode in \mathcal{W} reads as

$$X = v^{i} \frac{\partial}{\partial q^{i}} + F^{i} \frac{\partial}{\partial v^{i}} + G_{i} \frac{\partial}{\partial p_{i}} + f \frac{\partial}{\partial z}.$$

The bundle ${\mathcal W}$ is endowed with the following canonical structures:

Definition 8.

- The coupling function in W is the map C : W → R defined as follows: for every w = (v_q, p_q, z) ∈ W, where q ∈ Q, p_q ∈ T*Q, and v_q ∈ TQ, then C(w) := ⟨p_q, v_q⟩.
- 2. The canonical 1-form is the ρ_0 -semibasic form $\Theta := \rho_2^* \theta \in \Omega^1(W)$. The canonical 2-form is $\Omega := -d\Theta = \rho_2^* \omega \in \Omega^2(W)$.
- 3. The canonical contact 1-form is the ρ_1 -semibasic form $\eta := dz \Theta \in \Omega^1(W)$. Then $d\eta = \Omega$.

In natural coordinates of $\mathcal W$ we have that

 $\eta = \mathrm{d}z - p_i \,\mathrm{d}q^i, \quad \mathrm{d}\eta = \mathrm{d}q^i \wedge \mathrm{d}p_i.$

Definition 9. Given a Lagrangian function $L \in C^{\infty}(TQ \times \mathbb{R})$, let $\mathcal{L} = \rho_1^*L \in C^{\infty}(W)$. We define the Hamiltonian function

$$\mathcal{H} := \mathcal{C} - \mathcal{L} = p_i v^i - \mathcal{L}(q^j, v^j, z) \in C^{\infty}(\mathcal{W}).$$
(9)

Remark 8. Observe that η is a precontact form in W. Hence, (W, η) is a precontact manifold and (W, η, H) is a precontact Hamiltonian system.

As a consequence, Equations (1) do not have a unique solution and the Reeb vector fields are not uniquely defined. In fact, in natural coordinates of W the general solution to (1) are the vector fields $\mathcal{R} = \frac{\partial}{\partial z} + F^i \frac{\partial}{\partial v^i}$ for arbitrary coefficients F^i . Nevertheless, as we have pointed out, the formalism is independent on the choice of these Reeb vector fields. In our case, as $\mathcal{W} = TQ \times_Q T^*Q \times \mathbb{R}$ is a trivial bundle over \mathbb{R} , the canonical vector field $\frac{\partial}{\partial z}$ of \mathbb{R} can be lifted canonically to a vector field in \mathcal{W} , which can be taken as a representative of the family of Reeb vector fields.

3.2. Contact Dynamical Equations

Definition 10. The Lagrangian-Hamiltonian problem associated with the contact system $(\mathcal{W}, \eta, \mathcal{H})$ consists in finding the integral curves of a vector field $X_{\mathcal{H}} \in \mathfrak{X}(\mathcal{W})$ satisfying that $\mathfrak{b}(X_{\mathcal{H}}) = d\mathcal{H} - (\mathcal{R}(\mathcal{H}) + \mathcal{H})\eta$; that is, which is a solution to the contact Hamiltonian equations

$$i(X_{\mathcal{H}})d\eta = d\mathcal{H} - (\mathcal{R}(\mathcal{H}))\eta, \quad i(X_{\mathcal{H}})\eta = -\mathcal{H}.$$
(10)

or, what is equivalent,

$$L(X_{\mathcal{H}})\eta = -(\mathcal{R}(\mathcal{H}))\eta, \quad i(X_{\mathcal{H}})\eta = -\mathcal{H},$$

Then, the integral curves $\sigma : I \subset \mathbb{R} \to W$ of X_H , are the solutions to the equations

$$i(\sigma')d\eta = (d\mathcal{H} - (\mathcal{R}(\mathcal{H}))\eta)\circ\sigma, \quad i(\sigma')\eta = -\mathcal{H}\circ\sigma.$$
 (11)

As (W, η, H) is a precontact Hamiltonian system, these equations are not compatible everywhere in W, and we need to implement the standard constraint algorithm in order to find the final constraint submanifold of W (if it exists) where there are consistent solutions to the equations. Next we detail this procedure.

In a natural chart in \mathcal{W} , the local expression of a vector field $X_{\mathcal{H}} \in \mathfrak{X}(\mathcal{W})$ is

$$X_{\mathcal{H}} = f^{i} \frac{\partial}{\partial q^{i}} + F^{i} \frac{\partial}{\partial \nu^{i}} + G_{i} \frac{\partial}{\partial p_{i}} + f \frac{\partial}{\partial z}; \qquad (12)$$

and therefore we obtain that

$$i(X_{\mathcal{H}})\eta = f - f^i p_i ,$$

 $i(X_{\mathcal{H}})d\eta = f^i dp_i - G_i dq^i$

Furthermore,

$$\begin{split} \mathrm{d}\mathcal{H} &= \nu^{i}\mathrm{d}p_{i} + \left(p_{i} - \frac{\partial\mathcal{L}}{\partial\nu^{i}}\right)\mathrm{d}\nu^{i} - \frac{\partial\mathcal{L}}{\partial q^{i}}\,\mathrm{d}q^{i} - \frac{\partial\mathcal{L}}{\partial z}\,\mathrm{d}z \\ (\mathcal{R}(\mathcal{H}))\eta &= -\frac{\partial\mathcal{L}}{\partial z}(\mathrm{d}z - p_{i}\mathrm{d}q^{i}). \end{split}$$

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Then, the second Equation (10) gives

$$f = (f^i - v^i) p_i + \mathcal{L} , \qquad (13)$$

and the first Equation (10) leads to:

coefficients in
$$dp_i : f^i = v^i$$
, (14)

coefficients in
$$dv^i$$
: $p_i = \frac{\partial \mathcal{L}}{\partial v^i}$, (15)

coefficients in
$$dq^i$$
: $G_i = \frac{\partial \mathcal{L}}{\partial q^i} + p_i \frac{\partial \mathcal{L}}{\partial z}$, (16)

and the equalities from the coefficients in dz hold identically. From these equations, first we have that:

- The Equations (14) are the holonomy conditions (i.e., *X*_H is a sode). Thus, as it is usual, the sode condition arises straightforwardly from the unified formalism. This property reflects the fact that this geometric condition in the unified formalism is stronger than in the standard Lagrangian formalism.
- The algebraic Equations (15) are compatibility conditions defining a submanifold $W_1 \hookrightarrow W$, which is the *first constraint submanifold* of the Hamiltonian precontact system (W, η, H) , and is the graph of $\mathcal{F}L$; that is,

$$\mathcal{W}_1 = \{ (\mathbf{v}_q, \mathcal{F}L(\mathbf{v}_q)) \in \mathcal{W} \mid \mathbf{v}_q \in \mathrm{T}Q \}.$$

In this way, the unified formalism includes the definition of the Legendre map as a consequence of the constraint algorithm.

Therefore, vector fields solution to (10) are of the form

$$X_{\mathcal{H}} = \nu^{i} \frac{\partial}{\partial q^{i}} + F^{i} \frac{\partial}{\partial \nu^{i}} + \left(\frac{\partial \mathcal{L}}{\partial q^{i}} + p_{i} \frac{\partial \mathcal{L}}{\partial z}\right) \frac{\partial}{\partial p_{i}} + \mathcal{L} \frac{\partial}{\partial z} \quad (\text{on } \mathcal{W}_{1}).$$

Next, the constraint algorithm continues by demanding that $X_{\mathcal{H}}$ must be tangent to \mathcal{W}_1 , to ensure that dynamic trajectories remain in \mathcal{W}_1 . As $\xi_j^1 = p_j - \frac{\partial \mathcal{L}}{\partial v^j} \in C^{\infty}(\mathcal{W})$ are the constraints defining \mathcal{W}_1 , this condition is

$$X_{\mathcal{H}}\left(p_{j}-\frac{\partial\mathcal{L}}{\partial\nu^{j}}\right) = -\frac{\partial^{2}\mathcal{L}}{\partial q^{i}\partial\nu^{j}}\nu^{i} - \frac{\partial^{2}\mathcal{L}}{\partial\nu^{i}\partial\nu^{j}}F^{i}$$
$$-\mathcal{L}\frac{\partial^{2}\mathcal{L}}{\partial z\partial\nu^{j}} + \frac{\partial\mathcal{L}}{\partial q^{j}} + p_{j}\frac{\partial\mathcal{L}}{\partial z} = 0 \quad (\text{on } \mathcal{W}_{1}). \tag{17}$$

At this point we have to distinguish:

- If *L* is a regular Lagrangian, these equations allow us to determine all the functions $F^i = \frac{dy^i}{dt}$; then the solution is unique and the algorithm ends.
- If *L* is singular, then these equations establish relations among the arbitrary functions *Fⁱ*: some of them remain undetermined and the solutions are not unique. Eventually, new constraints ξ²_μ ∈ C[∞](W) can appear, defining a new submanifold W₂ → W₁ → W and then the algorithm continues by demanding that X_H must be tangent to W₂, and so on until we obtain a

final constraint submanifold \mathcal{W}_{f} (if it exists) where tangent solutions $X_{\mathcal{H}}$ exist.

Now, if $\sigma(t) = (q^i(t), v^i(t), p_i(t), z(t))$ is an integral curve of $X_{\mathcal{H}}$, we have that $f^i = \frac{dq^i}{dt}$, $F^i = \frac{dv^i}{dt}$, $G_i = \frac{dp_i}{dt}$, $f = \frac{dz}{dt}$, and then the Equations (13), (14), (15), and (16) lead to the coordinate expression of the Equations (11); in particular:

- From (14), we have that $v^i = \dot{q}^i$; that is, the holonomy condition.
- Using (14) again, the Equation (13) leads to

$$\dot{z} = \mathcal{L}$$
, (18)

which is just the Equation (7).

• The Equations (16) read

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q^i} + p_i \frac{\partial \mathcal{L}}{\partial z} = -\left(\frac{\partial \mathcal{H}}{\partial q^i} + p_i \frac{\partial \mathcal{H}}{\partial z}\right),$$

which are the second group of Hamilton's Equations (4). Then, using (15) (that is, on W_1), these equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) = \frac{\partial \mathcal{L}}{\partial q^i} + \frac{\partial \mathcal{L}}{\partial v^i}\frac{\partial \mathcal{L}}{\partial z}$$

which are the Euler-Lagrange Equations (8). The first group of Hamilton's Equations (4) arises straightforwardly from the definition of the Hamiltonian function (9), taking into account the holonomy condition.

Using (15) (that is, on W₁) and (18), the tangency condition (17) gives again the contact Euler-Lagrange Equations (8). Observe that, if L is singular, these equations could be incompatible.

3.3. Recovering the Lagrangian and Hamiltonian Formalisms and Equivalence

Next we study the equivalence of the unified formalism with the Lagrangian and Hamiltonian formalisms.

First, observe that, denoting by $J_1 : W_1 \hookrightarrow W$ the natural embedding, we have that

$$(\rho_1 \circ J_1)(\mathcal{W}_1) = \mathrm{T}Q \times \mathbb{R}, \quad (\rho_2 \circ J_1)(\mathcal{W}_1) = P_1 \subseteq \mathrm{T}^*Q \times \mathbb{R}.$$

In particular P_1 is a submanifold of $T^*Q \times \mathbb{R}$ when L is an almostregular Lagrangian (see Remark 7) and $P_1 = T^*Q \times \mathbb{R}$ when L is hyperregular (or an open set of $T^*Q \times \mathbb{R}$ if L is regular). Furthermore, as W_1 is the graph of the Legendre map $\mathcal{F}L$, it is diffeomorphic to $TQ \times \mathbb{R}$, being the restricted projection $\rho_1 o_{J_1}$ this diffeomorphism. In the same way, in the almost-regular case, for every submanifold $J_{\alpha} : W_{\alpha} \hookrightarrow W$ obtained by application of the constraint algorithm, we have

$$(\rho_1 \circ J_\alpha)(\mathcal{W}_\alpha) = S_\alpha \hookrightarrow \mathrm{T}Q \times \mathbb{R},$$
$$(\rho_1 \circ J_\alpha)(\mathcal{W}_\alpha) = P_\alpha \hookrightarrow P_1 \hookrightarrow \mathrm{T}^*O \times \mathbb{R},$$

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and, as $\mathcal{W}_{\alpha} \subseteq \mathcal{W}_{1} = \operatorname{graph} \mathcal{F}L$, then $\mathcal{F}L(S_{\alpha}) = P_{\alpha}$. Finally, let J_{f} : $\mathcal{W}_{f} \hookrightarrow \mathcal{W}$ the final constraint submanifold, and

$$(\rho_1 \circ J_f)(\mathcal{W}_{\alpha}) = S_f \hookrightarrow TQ \times \mathbb{R},$$

$$(\rho_2 \circ J_f)(\mathcal{W}_{\alpha}) = P_f \hookrightarrow P_1 \hookrightarrow T^*Q \times \mathbb{R}.$$

We have the diagram



Every function or differential form in \mathcal{W} and the vector fields in \mathcal{W} tangent to \mathcal{W}_1 can be restricted to \mathcal{W}_1 . Then, they can be translated to the Lagrangian or the Hamiltonian side by using that \mathcal{W}_1 is diffeomorphic to $TQ \times \mathbb{R}$, or projecting to the second factors of the product bundle, $T^*Q \times \mathbb{R}$. Therefore, bearing this in mind, the results and the discussion in the above section lead to state:

Theorem 2. Every path $\sigma : I \subseteq \mathbb{R} \to W$, taking values in W_1 , can be split as $\sigma = (\sigma_L, \sigma_H)$, where $\sigma_L = \rho_1 \circ \sigma : I \subseteq \mathbb{R} \to TQ \times \mathbb{R}$ and $\sigma_H = \mathcal{F}L \circ \sigma_L : I \subseteq \mathbb{R} \to P_1 \subseteq T^*Q \times \mathbb{R}$.

Let $\boldsymbol{\sigma} : \mathbb{R} \to \mathcal{W}$, with Im $(\boldsymbol{\sigma}) \subset \mathcal{W}_1$, be a path fulfilling the Equations (11) (at least on the points of a submanifold $\mathcal{W}_f \hookrightarrow \mathcal{W}_1$). Then $\boldsymbol{\sigma}_L$ is the prolongation to $TQ \times \mathbb{R}$ of the projected curve $\mathbf{c} = \rho_0 \circ \boldsymbol{\sigma}$: $\mathbb{R} \to Q \times \mathbb{R}$ (that is, $\boldsymbol{\sigma}_L$ is a holonomic section), and it is a solution to the Equations (5). Moreover, the path $\boldsymbol{\sigma}_H = \mathcal{F} L \circ \widetilde{\mathbf{c}}$ is a solution to the Equations (3) (on \mathcal{W}_f).

Conversely, for every path $\mathbf{c} : \mathbb{R} \to Q \times \mathbb{R}$ such that $\mathbf{\tilde{c}}$ is a solution to the Equation (5) (on S_f), we have that the section $\boldsymbol{\sigma} = (\mathbf{\tilde{c}}, \mathcal{F}L \circ \mathbf{\tilde{c}})$ is a solution to the Equations (11). Furthermore, $\mathcal{F}L \circ \mathbf{\tilde{c}}$ is a solution to the Equation (3)) (on P_f).

Notice that, if *L* is a singular Lagrangian, then these results hold on the points of the submanifolds W_f , S_f and P_f

As the paths $\boldsymbol{\sigma} : \mathbb{R} \to \mathcal{W}$ solution to the Equation (11) are the integral curves of holonomic vector fields $X_{\mathcal{H}} \in \mathfrak{X}(\mathcal{W})$ solution to (10), and the paths $\boldsymbol{\sigma}_L : \mathbb{R} \to TQ \times \mathbb{R}$ are the integral curves of holonomic vector fields $X_L \in \mathfrak{X}(TQ \times \mathbb{R})$ solution to (5), then an immediate corollary of the above theorem is:

Theorem 3. Let $X_{\mathcal{H}} \in \mathfrak{X}(\mathcal{W})$ be a vector field which is solution to the Equations (10) (at least on the points of a submanifold $\mathcal{W}_{f} \hookrightarrow \mathcal{W}_{1}$)

and tangent to W_1 (resp. tangent to W_f). Then the vector field $X_L \in \mathfrak{X}(\mathrm{TQ} \times \mathbb{R})$, defined by $X_L \circ \rho_1 = \mathrm{T\rho}_1 \circ X_H$, is a holonomic vector field (tangent to S_f) which is a solution to the Equations (6) (on S_f), where $\mathcal{H} = \rho_1^* E_I$.

In addition, every holonomic vector field solution to the Equations (6) (on S_f) can be recovered in this way from a vector field $X_H \in \mathfrak{X}(W)$ (tangent to W_f) solution to the Equations (10) (on W_f).

The Hamiltonian formalism is recovered in a similar way, taking into account that, now, the paths $\sigma_H : \mathbb{R} \to \mathrm{T}^*Q \times \mathbb{R}$ are the integral curves of vector fields $X_H \in \mathfrak{X}(\mathrm{T}^*Q \times \mathbb{R})$ solution to (2). So we have:

Theorem 4. Let $X_{\mathcal{H}} \in \mathfrak{X}(\mathcal{W})$ be a vector field which is solution to the Equations (10) (at least on the points of a submanifold $\mathcal{W}_f \hookrightarrow \mathcal{W}_1$) and tangent to \mathcal{W}_1 (resp. tangent to \mathcal{W}_f). Then the vector field $X_H \in \mathfrak{X}(T^*Q \times \mathbb{R})$, defined by $X_H \circ \rho_2 = T\rho_2 \circ X_H$, is a solution to the Equations (2) (on P_f and tangent to P_f), where $\mathcal{H} = \rho_2^* \mathcal{H}$.

Remark 9. These results are the same that those obtained for the unified formalism of non-autonomous dynamical systems. Intrinsic proofs of the corresponding theorems can be found in [3] (see also [9]).

Remark 10. It is important to point out that, when working with singular Lagrangians, the equivalence between the constraint algorithms in the unified and in the Lagrangian formalism only holds when the holonomy (or second-order) condition is imposed as an additional condition for the solutions in the Lagrangian case since, unlike in the unified formalism, this condition does not hold in the Lagrangian case (see [39, 47]).

4. Examples

4.1. General Features

In the following examples we consider some dynamical systems described by Lagrangians which have been modified by adding a term of dissipation.^[15,27] So, we consider the following situation. Let Q be an *n*-dimensional differentiable manifold and let $L = \tau_1^* L_o - \gamma z \in \mathbb{C}^{\infty}(TQ \times \mathbb{R})$ be a Lagrangian, where $\gamma \in \mathbb{R}$ and $L_0 \in \mathbb{C}^{\infty}(TQ)$ is a either a regular or a singular Lagrangian. Let $\mathcal{W} = TQ \times_Q T^*Q \times \mathbb{R}$ be the extended unified bundle, with local coordinates (q^i, v^i, p_i, z) , and denote $\mathcal{L} = \rho_1^* L \in \mathbb{C}^{\infty}(\mathcal{W})$ which is a regular or singular Lagrangian depending on the regularity of L_o (in the singular case, we assume that it is almost-regular). Then

$$\mathcal{H} = p_i v^i - L_o(q^i, v^i) + \gamma z \in C^{\infty}(\mathcal{W})$$

and

$$\mathrm{d}\mathcal{H} = \nu^{i}\mathrm{d}p_{i} + \left(p_{i} - \frac{\partial L_{o}}{\partial\nu^{i}}\right)\mathrm{d}\nu^{i} - \frac{\partial L_{o}}{\partial q^{i}}\,\mathrm{d}q^{i} + \gamma\,\mathrm{d}z.$$

Now, for a vector field $X_{\mathcal{H}} \in \mathfrak{X}(\mathcal{W})$ with local expression (12), the Equations (10) give

$$\begin{split} f^{i} &= v^{i} \text{ , } f = (f^{i} - v^{i}) p_{i} + \mathcal{L} = \mathcal{L} \\ p_{i} &= \frac{\partial L_{o}}{\partial v^{i}} \text{ , } G_{i} = \frac{\partial L_{o}}{\partial q^{i}} - \gamma p_{i}. \end{split}$$

We have the submanifold $\mathcal{W}_1 = \operatorname{graph}(\mathcal{F}L) \hookrightarrow \mathcal{W}$, and

$$X_{\mathcal{H}}\Big|_{W_1} = \nu^i \frac{\partial}{\partial q^i} + F^i \frac{\partial}{\partial \nu^i} + \left(\frac{\partial L_o}{\partial q^i} - \gamma p_i\right) \frac{\partial}{\partial p_i} + (L_o - \gamma z) \frac{\partial}{\partial z}.$$

The tangency condition of $X_{\mathcal{H}}$ to \mathcal{W}_1 leads to

$$X_{\mathcal{H}}\left(p_{j}-\frac{\partial L_{o}}{\partial v^{j}}\right) = -\frac{\partial^{2} L_{o}}{\partial q^{i} \partial v^{j}} v^{i} - \frac{\partial^{2} L_{o}}{\partial v^{i} \partial v^{j}} F^{i} + \frac{\partial L_{o}}{\partial q^{j}} - \gamma p_{j} = 0$$

(on \mathcal{W}_{1}).

As remarked in Section 3.2, if the Lagrangian is regular, these equations allows us to determine all the coefficients F^i and we have a unique solution. In the singular case, these equations establish relations among the arbitrary functions F^i and, eventually, new constraints could appear, defining a new submanifold $\mathcal{W}_2 \hookrightarrow \mathcal{W}_1 \hookrightarrow \mathcal{W}$. Then, the algorithm continues until we obtain a final constraint submanifold \mathcal{W}_f (if it exists) where tangent solutions $X_{\mathcal{H}}$ exist.

If $\sigma(t) = (q^i(t), v^i(t), p_i(t), z(t))$ is an integral curve of a solution $X_{\mathcal{H}}$ tangent to \mathcal{W}_f , the Equations (11), on the points of \mathcal{W}_f , are in this case

$$\begin{split} \dot{z} &= L_o - \gamma z, \quad \dot{q}^i = v^i, \\ \dot{p}^i &= \frac{d}{dt} \left(\frac{\partial L}{\partial v^i} \right) = \frac{\partial L_o}{\partial q^i} - \gamma p_i = \frac{\partial L_o}{\partial q^i} - \gamma \frac{\partial L_o}{\partial v^i}. \end{split}$$

Next we analyze three examples: one regular system and two singular cases, one with a unique solution and the other with multiple solutions.

4.2. Regular Example: Central Force with Dissipation

Consider the system made of a particle in \mathbb{R}^3 with mass m, submitted to a central potential with dissipation. Taking $Q = \mathbb{R}^3 - \{(0, 0, 0)\}$ with local coordinates (q^i) , the Lagrangian that describes the dynamics is

$$L = \frac{1}{2}m v_i v^i - U(r) - \gamma z \in C^{\infty}(TQ \times \mathbb{R}) ,$$

where $v_i = g_{ij}v^j$, being g_{ij} the natural extension to \mathcal{W} of the euclidean metric in \mathbb{R}^3 , and $r = \sqrt{q_i q^i}$. In the extended unified bundle $\mathcal{W} = TQ \times_Q T^*Q \times \mathbb{R}$, with local coordinates (q^i, v^i, p_i, z) , we denote $\mathcal{L} = \rho_1^* L \in \mathbb{C}^{\infty}(\mathcal{W})$, which has the same coordinate expression that *L* and is a hyperregular Lagrangian. Then

$$\mathcal{H} = p_i v^i - \frac{1}{2} m v_i v^i + U(r) + \gamma z \in C^{\infty}(\mathcal{W}) ,$$

and

$$\mathrm{d}\mathcal{H} = v^i \mathrm{d}p_i + (p_i - m \, v_i) \mathrm{d}v^i + \frac{U'(r)}{r} q^i \, \mathrm{d}q^i + \gamma \, \mathrm{d}z.$$

Now, for a vector field $X_{\mathcal{H}} \in \mathfrak{X}(\mathcal{W})$, whose local expression is (12), the Equations (10) give

$$\begin{split} f^{i} &= \nu^{i}, \quad f = (f^{i} - \nu^{i}) \, p_{i} + \mathcal{L} = \mathcal{L} , \\ p_{i} &= m \, \nu_{i}, \quad G_{i} = -\frac{U'(r)}{r} q_{i} - \gamma \, p_{i}. \end{split}$$

Thus we have the submanifold $\mathcal{W}_1 \hookrightarrow \mathcal{W}$ defined by

$$\mathcal{W}_1 = \{(q^i, v^i, p_i, z) \in \mathcal{W} \mid p_i - m v_i = 0\} = \operatorname{graph}(\mathcal{F}L).$$

and

$$\begin{split} X_{\mathcal{H}}\Big|_{\mathcal{W}_{1}} &= \nu^{i} \frac{\partial}{\partial q^{i}} + F^{i} \frac{\partial}{\partial \nu^{i}} - \left(\gamma p_{i} + \frac{U'(r)}{r} q_{i}\right) \frac{\partial}{\partial p_{i}} \\ &+ \left(\frac{1}{2} m \nu_{i} \nu^{i} - U(r) - \gamma z\right) \frac{\partial}{\partial z}. \end{split}$$

Next, the tangency condition of $X_{\mathcal{H}}$ to \mathcal{W}_1 leads to

$$\begin{aligned} X_{\mathcal{H}}(p_i - m \, \nu_i) &= -\gamma \, p_i - \frac{U'(r)}{r} q_i - m \, F_i = 0 \iff F^i \\ &= -\frac{1}{m} \left(\gamma \, p^i + \frac{U'(r)}{r} q^i \right) \quad (\text{on } \mathcal{W}_1), \end{aligned}$$

and the algorithm finishes giving the unique solution

$$\begin{split} X_{\mathcal{H}}\Big|_{\mathcal{W}_{1}} &= v^{i}\frac{\partial}{\partial q^{i}} - \frac{1}{m}\bigg(\gamma p^{i} + \frac{U'(r)}{r}q^{i}\bigg)\frac{\partial}{\partial v^{i}} \\ &-\bigg(\gamma p_{i} + \frac{U'(r)}{r}q_{i}\bigg)\frac{\partial}{\partial p_{i}} + \mathcal{L}\frac{\partial}{\partial z}. \end{split}$$

Therefore, if $\sigma(t) = (q^i(t), v^i(t), p_i(t), z(t))$ is an integral curve of $X_{\mathcal{H}}$, the Equations (11), on the points of \mathcal{W}_1 , are

$$\dot{z} = \mathcal{L}, \quad \dot{q}^{i} = v^{i}, \quad \frac{1}{m} \dot{p}^{i} = \dot{v}^{i} = \ddot{q}^{i} = -\gamma \, \dot{q}^{i} - \frac{U'(r)}{m \, r} q^{i};$$

which are the Euler-Lagrange equations for the motion of a particle in a central potential with friction.

As stated in Section 3.3, we can recover the Lagrangian and the Hamiltonian formalisms by projecting on each factor of $\mathcal{W} = TQ \times_Q T^*Q \times \mathbb{R}$. In this case, as *L* is a hyperregular Lagrangian, $\mathcal{F}L : TQ \times \mathbb{R} \to T^*Q \times \mathbb{R}$ is a diffeormorphism, and the constraint algorithm finishes in the manifold \mathcal{W}_1 . Then, in the Lagrangian formalism, we have the holonomic contact Lagrangian vector field

$$\begin{split} X_{L} &= \nu^{i} \frac{\partial}{\partial q^{i}} - \left(\gamma \, \nu_{i} + \frac{U'(r)}{mr} q^{i}\right) \frac{\partial}{\partial \nu^{i}} \\ &+ \left(\frac{1}{2}m \, \nu_{i} \nu^{i} - U(r) - \gamma z\right) \frac{\partial}{\partial z} \in \mathfrak{X}(\mathrm{T}Q \times \mathbb{R}), \end{split}$$

and, in the Hamiltonian formalism, we have the contact Hamiltonian vector field

$$\begin{split} X_{H} &= \frac{p_{i}}{m} \frac{\partial}{\partial q^{i}} - \left(\gamma p_{i} + \frac{U'(r)}{r} q_{i}\right) \frac{\partial}{\partial p_{i}} \\ &+ \left(\frac{p_{i} p^{i}}{2m} - U(r) - \gamma z\right) \frac{\partial}{\partial z} \in \mathfrak{X}(\mathrm{T}^{*}Q \times \mathbb{R}). \end{split}$$

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4.3. Singular Example: Lagrange Multipliers (the Damped Simple Pendulum)

The *Lagrange multipliers method* to incorporate constraints in a system leads to singular Lagrangians in a natural way, since the velocities of the multipliers do not appear in the Lagrangian. In order to expose how to apply this formalism to system with Lagrange multipliers, we present a simple case: the pendulum under gravity with air friction.

Consider a pendulum with mass *m* and length *l*. Its position in the plain of motion is given by the polar coordinates (*r*, θ), such that $\theta = 0$ while at rest. This motion is restricted to the circumference *r* = *l*. The corresponding Lagrangian is

$$L = \frac{1}{2}m(v_r^2 + r^2 v_\theta^2) - mgr(1 - \cos\theta) + \lambda(r - l)$$
$$-\gamma z \in C^{\infty}(T\mathbb{R}^3 \times \mathbb{R}).$$

where λ is the Lagrange multiplier and we have added a dissipative term $-\gamma z$. It is a singular Lagrangian since the generalized velocity ν^{λ} does not appear in the Lagrangian. In the extended unified bundle $\mathcal{W} = \mathbb{TR}^3 \times_{\mathbb{R}^3} \mathbb{T}^*\mathbb{R}^3 \times \mathbb{R}$, with local coordinates $(r, \theta, \lambda, \nu_r, \nu_{\theta}, \nu_{\lambda}, p_r, p_{\theta}, p_{\lambda}, z)$, we denote $\mathcal{L} = \rho_1^* L \in \mathbb{C}^{\infty}(\mathcal{W})$, which has the same coordinate expression that *L*. Then

$$\begin{aligned} \mathcal{H} &= p_r v_r + p_\theta v_\theta + p_\lambda v_\lambda - \frac{1}{2} m (v_r^2 + r^2 v_\theta^2) + mgr(1 - \cos\theta) \\ &+ \gamma z - \lambda (r - l) \in \mathbf{C}^{\infty}(\mathcal{W}). \end{aligned}$$

Now, for a vector field $X_{\mathcal{H}} \in \mathfrak{X}(\mathcal{W})$, whose local expression is (12), the Equations (10) give

$$\begin{split} f &= \mathcal{L}, \\ f_r &= v_r, \quad f_\theta = v_\theta, \quad f_\lambda = v_\lambda, \\ p_r &= mv_r, \quad p_\theta = r^2 mv_\theta, \quad p_\lambda = 0, \\ G_r &= mrv_\theta^2 - mg(1 - \cos\theta) + \lambda - \gamma p_r, \\ G_\theta &= -mgr\sin\theta - \gamma p_\theta, \quad G_\lambda = r - l - \gamma p_\lambda. \end{split}$$

Thus we have the submanifold $\mathcal{W}_1 \hookrightarrow \mathcal{W}$ defined by

$$\begin{split} \mathcal{W}_1 &= \{(r,\theta,\lambda,v_r,v_\theta,v_\lambda,p_r,p_\theta,p_\lambda,z) \in \mathcal{W} \mid p_r = mv_r, \\ p_\theta &= mr^2 v_\theta, \quad p_\lambda = 0\} = \text{graph}(\mathcal{F}L), \end{split}$$

and the vector field

$$\begin{split} X_{\mathcal{H}}\Big|_{\mathcal{W}_{1}} &= \mathcal{L}\frac{\partial}{\partial z} + \nu_{r}\frac{\partial}{\partial r} + \nu_{\lambda}\frac{\partial}{\partial \lambda} + \nu_{\theta}\frac{\partial}{\partial \theta} + F_{r}\frac{\partial}{\partial \nu_{r}} + F_{\theta}\frac{\partial}{\partial \nu_{\theta}} + F_{\lambda}\frac{\partial}{\partial \nu_{\lambda}} \\ &+ (mrv_{\theta}^{2} - mg(1 - \cos\theta) + \lambda - \gamma p_{r})\frac{\partial}{\partial p_{r}} \\ &- (mgr\sin\theta + \gamma p_{\theta})\frac{\partial}{\partial p_{\theta}} + (r - l - \gamma p_{\lambda})\frac{\partial}{\partial p_{\lambda}}. \end{split}$$

The tangency condition of $X_{\mathcal{H}}$ to \mathcal{W}_1 leads to

$$F_r = rv_{\theta}^2 - g(1 - \cos \theta) + \frac{\lambda}{m} - \gamma v_r,$$

$$2v_r v_{\theta} + rF_{\theta} = -g \sin \theta - \gamma r v_{\theta}, \quad r = l \text{ (on } \mathcal{W}_1\text{)}$$
(19)

So, we recover dynamically the constraint r = l. The tangency condition to the submanifold W_2 defined by all these constraints gives

$$v_r = 0$$
 (on \mathcal{W}_2).

Imposing again the tangency condition on the new submanifold W_3 so obtained, we obtain a new equation $F_r = 0$, which allows us to compute the Lagrange multiplier

$$\lambda = mg(1 - \cos\theta) - mlv_{\theta}^2 \quad (\text{on } \mathcal{W}_3).$$

This is a new constraint, and we have the submanifold W_4 , where the tangency condition leads to obtain a last constraint

$$v_{\lambda} = m(3gv_{\theta}\sin\theta + 2l\gamma v_{\theta}^2) \quad (\text{on } \mathcal{W}_4)$$

Finally, the tangency condition on this constraint allows us to determine

$$F_{\lambda} = mg \left(3v_{\theta} \cos \theta - 3\frac{g}{l} \sin^2 \theta - 5\gamma v_{\theta} \sin \theta - 2lg v_{\theta}^2 \right) \quad (\text{on } \mathcal{W}_4) ,$$

and the algorithm finishes with the final constraint submanifold $W_f = W_4$, which is defined as

$$\mathcal{W}_{f} = \{ (r, \theta, \lambda, \nu_{r}, \nu_{\theta}, \nu_{\lambda}, p_{r}, p_{\theta}, p_{\lambda}, z) \in \mathcal{W} \mid p_{r} = m\nu_{r}, \\ p_{\theta} = mr^{2}\nu_{\theta}, \quad p_{\lambda} = 0, \quad r = l, \quad \nu_{r} = 0, \\ \lambda = mg(1 - \cos\theta) - ml\nu_{\theta}^{2}, \quad \nu_{\lambda} = m(3g\nu_{\theta}\sin\theta + 2l\gamma\nu_{\theta}^{2}) \}$$

and the unique solution

$$\begin{split} X_{\mathcal{H}}\Big|_{\mathcal{W}_{f}} &= m(3gv_{\theta}\sin\theta + 2l\gamma v_{\theta}^{2})\frac{\partial}{\partial\lambda} + v_{\theta}\frac{\partial}{\partial\theta} - \left(\frac{g}{l}\sin\theta + \gamma v_{\theta}\right)\frac{\partial}{\partial v_{\theta}} \\ &+ mg\Big(3v_{\theta}\cos\theta - 3\frac{g}{l}\sin^{2}\theta - 5\gamma v_{\theta}\sin\theta - 2lgv_{\theta}^{2}\Big)\frac{\partial}{\partial v_{\lambda}} \\ &- ml(g\sin\theta + \gamma lv_{\theta})\frac{\partial}{\partial p_{\theta}} \\ &+ \Big(\frac{1}{2}ml^{2}v_{\theta}^{2} - mgl(1 - \cos\theta) - \gamma z\Big)\frac{\partial}{\partial z}. \end{split}$$

Observe that there are only three independent variables: z, θ , and v_{θ} . Therefore, for an integral curve of $X_{\mathcal{H}}$, the second equation of (19), on \mathcal{W}_f , gives the equation of motion for the only physical degree of freedom,

$$\ddot{\theta} = -\frac{g}{l}\sin\theta - \gamma\dot{\theta} ;$$

which is the usual equation of the damped simple pendulum.

As stated above, we can recover the Lagrangian and the Hamiltonian formalisms by projecting on each factor of $\mathcal{W} = T\mathbb{R}^3 \times_{\mathbb{R}^3} T^*\mathbb{R}^3 \times \mathbb{R}$. Thus, in the Lagrangian formalism, we have the final constraint submanifold

$$S_{f} = \{ (r, \theta, \lambda, v_{r}, v_{\theta}, v_{\lambda}, z) \in \mathcal{W} \mid r = l, \quad v_{r} = 0, \\ \lambda = mg(1 - \cos\theta) - mlv_{\theta}^{2}, \quad v_{\lambda} = m(3gv_{\theta}\sin\theta + 2l\gamma v_{\theta}^{2}) \}$$

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and the holonomic contact Lagrangian vector field

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$$\begin{split} X_L \Big|_{S_f} &= v_\theta \frac{\partial}{\partial \theta} + v_\lambda \frac{\partial}{\partial \lambda} - \left(\frac{g}{l}\sin\theta + \gamma v_\theta\right) \frac{\partial}{\partial v_\theta} + F_\lambda \frac{\partial}{\partial v^\lambda} \\ &+ \left(\frac{1}{2}ml^2 v_\theta^2 - mgl(1 - \cos\theta) - \gamma z\right) \frac{\partial}{\partial z} \in \mathfrak{X}(\mathbb{TR}^3 \times \mathbb{R}) \end{split}$$

Furthermore, in the Hamiltonian formalism, we have

$$P_{f} = \left\{ (r, \theta, \lambda, p_{r}, p_{\theta}, p_{\lambda}, z) \in \mathrm{T}^{*} \mathbb{R}^{3} \times \mathbb{R} \mid r = l, \quad p_{\lambda} = 0, \\ p_{r} = 0, \quad \lambda = mg(1 - \cos \theta) - \frac{p_{\theta}^{2}}{ml^{3}} \right\}$$

and the contact Hamiltonian vector field

$$\begin{split} X_{H}\Big|_{P_{f}} &= \frac{p_{\theta}}{ml^{2}} \frac{\partial}{\partial \theta} + \left(\frac{3g}{l^{2}} p_{\theta} \sin \theta + \frac{2\gamma}{ml^{3}} p_{\theta}^{2}\right) \frac{\partial}{\partial \lambda} \\ &- \left(mlg \sin \theta + \gamma p_{\theta}\right) \frac{\partial}{\partial p_{\theta}} \\ &+ \left(\frac{p_{\theta}^{2}}{2ml^{2}} - mgl(1 - \cos \theta) - \gamma z\right) \frac{\partial}{\partial z} \in \mathfrak{X}(\mathrm{T}^{*}\mathbb{R}^{3} \times \mathbb{R}) \end{split}$$

4.4. Singular Example: Cawley's Lagrangian with Dissipation

The last example is an academic model based on a known Lagrangian introduced by R. Cawley to study some characteristic features of singular Lagrangians in Dirac's theory of constrained systems.^[14]

In TR³ × R, with local coordinates (q^i , v^i , z), i = 1, 2, 3, consider the Lagrangian

$$L = v^{1}v^{3} + \frac{1}{2}q^{2}(q^{3})^{2} - \gamma z.$$

In the extended unified bundle $\mathcal{W} = \mathbb{TR}^3 \times_{\mathbb{R}^3} \mathbb{T}^*\mathbb{R}^3 \times \mathbb{R}$, with local coordinates (q^i, v^i, p_i, z) , we denote $\mathcal{L} = \rho_1^* L \in \mathbb{C}^{\infty}(\mathcal{W})$, which has the same coordinate expression that *L*. Then

$$\mathcal{H} = p_i \nu^i - \nu^1 \nu^3 - \frac{1}{2} q^2 (q^3)^2 + \gamma z \in \mathsf{C}^{\infty}(\mathcal{W}).$$

Now, for a vector field $X_{\mathcal{H}} \in \mathfrak{X}(\mathcal{W})$, with local expression (12), the Equations (10) give

$$\begin{aligned} p_1 &= v_3, \quad p_2 = 0, \quad p_3 = v_1, \\ f &= \mathcal{L}, \quad f^i = v^i, \quad G_1 = -\gamma p_1, \quad G_2 = \frac{1}{2}q^3 - \gamma p_2, \\ G_3 &= q^2q^3 - \gamma p_3. \end{aligned}$$

Thus we have the submanifold defined by

$$\mathcal{W}_1 = \{ (q^i, v^i, p_i, z) \in \mathcal{W} \mid p_1 = v_3, \quad p_2 = 0, \quad p_3 = v_1 \}$$
$$= \operatorname{graph}(\mathcal{F}L) \hookrightarrow \mathcal{W},$$

and the vector fields

$$\begin{split} X_{\mathcal{H}}\Big|_{\mathcal{W}_{1}} &= \nu^{i}\frac{\partial}{\partial q^{i}} + F^{i}\frac{\partial}{\partial \nu^{i}} - \gamma p_{1}\frac{\partial}{\partial p_{1}} + \frac{1}{2}q^{3}\frac{\partial}{\partial p_{2}} \\ &+ (q^{2}q^{3} - \gamma p_{3})\frac{\partial}{\partial p_{3}} + \mathcal{L}\frac{\partial}{\partial z}. \end{split}$$

The tangency condition of X_H to W_1 leads to determine F_1 and F_3 and gives a new constraint,

$$F_1 = q^2 q^3 - \gamma p_3$$
, $F_3 = -\gamma p_1$, $q^3 = 0$ (on \mathcal{W}_1).

Imposing the tangency condition on the submanifold W_2 defined by all these constraints we obtain

$$v^3 = 0$$
 (on \mathcal{W}_2),

which, bearing in mind the first constraint $p_1 = v_3$, implies that $p_1 = 0$ (on W_2). At this point, the tangency condition holds and we have the final constraint submanifold

$$\mathcal{W}_{f} = \{(q^{i}, v^{i}, p_{i}, z) \in \mathcal{W} \mid p_{1} = v_{3} = 0, p_{2} = 0, p_{3} = v_{1}, q^{3} = 0\}$$

and the family of solutions

$$X_{\mathcal{H}}\Big|_{W_{f}} = v^{1} \frac{\partial}{\partial q^{1}} + v^{2} \frac{\partial}{\partial q^{2}} - \gamma v_{1} \frac{\partial}{\partial v^{1}} + F^{2} \frac{\partial}{\partial v^{2}} - \gamma z \frac{\partial}{\partial z}$$

As in the above examples, we can recover the Lagrangian and the Hamiltonian formalisms by projecting on each factor of $\mathcal{W} = T\mathbb{R}^3 \times_{\mathbb{R}^3} T^*\mathbb{R}^3 \times \mathbb{R}$. Then, in the Lagrangian formalism, we have the final constraint submanifold

$$S_f = \{(q^i, v^i, z) \in \mathbb{TR}^3 \times \mathbb{R} \mid q^3 = 0, v^3 = 0\}$$

and the holonomic contact Lagrangian vector fields

$$X_L\Big|_{S_f} = \nu^1 \frac{\partial}{\partial q^1} + \nu^2 \frac{\partial}{\partial q^2} - \gamma \nu_1 \frac{\partial}{\partial \nu^1} + F^2 \frac{\partial}{\partial \nu^2} - \gamma z \frac{\partial}{\partial z} \in \mathfrak{X}(\mathbb{T}\mathbb{R}^3 \times \mathbb{R}).$$

Furthermore, in the Hamiltonian formalism, we have

$$P_{f} = \{ (q^{i}, p_{i}, z) \in T^{*} \mathbb{R}^{3} \times \mathbb{R} \mid p_{1} = 0, p_{2} = 0, q^{3} = 0 \}$$

and the unique contact Hamiltonian vector field

$$X_H\Big|_{P_f} = p_3 \frac{\partial}{\partial q^1} + \nu^2 \frac{\partial}{\partial q^2} - \gamma p_1 \frac{\partial}{\partial p_1} - \gamma z \frac{\partial}{\partial z} \in \mathfrak{X}(\mathbb{T}^* \mathbb{R}^3 \times \mathbb{R}) ,$$

(observe that ker $\mathcal{F}L = \langle \frac{\partial}{\partial v^2} \rangle$).

5. Conclusion and Outlook

We have presented a generalized framework for describing both the Lagrangian and the Hamiltonian formalism for autonomous contact dynamical systems. The key tool consists in using the natural geometric structure of the manifold $W = TQ \times_Q T^*Q \times \mathbb{R}$ (the *unified* or *Pontryagin bundle*) to define a precontact dynamical system, starting from a regular or an almost-regular Lagrangian function *L* in $TQ \times \mathbb{R}$. The compatibility of the dynamical equations stated in \mathcal{W} leads to define a submanifold \mathcal{W}_1 which is identified with the graph of the Legendre map $\mathcal{F}L$. As in other situations, the contact dynamical equations in the unified formalism are of three classes, giving different kinds of information:

- Algebraic (not differential) equations, which, in coordinates, read $p_i = \frac{\partial L}{\partial v^i}$, and determine the submanifold W_1 of W where the sections solution to the dynamical equations must take their values. For singular Lagrangians, the constraints defining W_1 , projected by ρ_2 , give the primary constraints of the Hamiltonian formalism; that is, The ρ_2 -projection of W_1 is the image of the Legendre transformation.
- The holonomic conditions, which in coordinates are $v^i = \frac{dq^i}{dt}$. These conditions force the dynamical trajectories to be holonomic curves. This property, which arise straightforwardly from the dynamical equations in the unified formalism, reflects the fact that, in the unified formalism, the second-order condition is stronger than the in the standard Lagrangian formalism.
- The contact Euler–Lagrange equations or, equivalently, the contact Hamiltonian equations.

As we have a precontact dynamical system, a constraint algorithm must be implemented in order to obtain a final constraint submanifold $W_f \hookrightarrow W_1$ where there are consistent solutions to the contact equations (i.e., trajectories tangent to W_f). As in the standard unified formalisms, if *L* is regular, then $W_f = W_1$. This algorithm is related (through the natural projections) with the corresponding ones in the Lagrangian and the Hamiltonian sides; although in the Lagrangian case, this equivalence only holds when the second-order condition is imposed as an additional condition for the solutions.

In addition, we have also discussed several interesting examples that illustrate the behaviour of the algorithm in the regular and singular cases.

The formalism stated here could serve as a starting point to set the unified formalism for k-contact systems in nonconservative field theories,^[25,26] as well as in other physical systems involving contact structures.

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Conflict of Interest

The authors have declared no conflict of interest.

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